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# Factorization of the hypergeometric-type difference equation on non-uniform lattices: dynamical algebra

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## Abstract

We argue that one can factorize the difference equation of hypergeometric type on non-uniform lattices in the general case. It is shown that in the most cases of  $q$ -linear spectrum of the eigenvalues, this directly leads to the dynamical symmetry algebra  $su_q(1, 1)$ , whose generators are explicitly constructed in terms of the difference operators, obtained in the process of factorization. Thus all models with the  $q$ -linear spectrum (some of them, but not all, previously considered in a number of publications) can be treated in a unified form.

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## 1. Introduction and preliminaries

In this paper we continue the study, started in [1], on the factorization of the hypergeometric-type difference equation on non-uniform lattices, i.e., of the equation [2]

$$\sigma(s) \frac{\Delta}{\Delta x \left(s - \frac{1}{2}\right)} \left[ \frac{\nabla y(s)}{\nabla x(s)} \right] + \tau(s) \frac{\Delta y(s)}{\Delta x(s)} + \lambda y(s) = 0, \quad (1)$$

$$\sigma(s) = \tilde{\sigma}(x(s)) - \frac{1}{2} \tilde{\tau}(x(s)) \Delta x \left(s - \frac{1}{2}\right), \quad \tau(s) = \tilde{\tau}(x(s)),$$

where  $\Delta y(s) := y(s+1) - y(s)$ ,  $\nabla y(s) := y(s) - y(s-1)$ ,  $\tilde{\sigma}(x(s))$  and  $\tilde{\tau}(x(s))$  are polynomials in  $x(s)$  of degree at most 2 and 1, respectively, and  $\lambda$  is a constant (see also [3]). The difference equation (1) has polynomial solutions  $P_n(x(s); q) := P_n(s; q)$  of hypergeometric type if and only if the lattice  $x(s)$  has the form [4, 5]

$$x(s) = c_1(q)q^s + c_2(q)q^{-s} + c_3(q) = c_1(q)[q^s + q^{-s-\mu}] + c_3(q), \quad (2)$$

where  $c_1, c_2, c_3$  and  $q^\mu := c_1/c_2$  are constants which, in general, depend on  $q$ . An important special case of the lattice  $x(s)$  is the  $q$ -linear lattice, which is obtained from (2) by assuming that either  $c_1(q)$  or  $c_2(q)$  vanishes.

The polynomial solutions of the difference equation (1) correspond to the following expression [3] for its eigenvalues  $\lambda_n(q)$ :

$$\begin{aligned} \lambda_n(q) &= C_1 q^n + C_2 q^{-n} + C_3, \\ C_1 &= \frac{1}{2(1-q)} \left( \tilde{\tau}' + \frac{\tilde{\sigma}''}{k_q} \right), \quad C_2 = \frac{1}{2(1-q^{-1})} \left( \tilde{\tau}' - \frac{\tilde{\sigma}''}{k_q} \right), \\ C_3 &= -\frac{\tilde{\sigma}''(1+q)}{2k_q(1-q)} - \frac{\tilde{\tau}'}{2}, \end{aligned} \quad (3)$$

where  $\tilde{\tau}'$  and  $\tilde{\sigma}''/2$  are the coefficients of  $x(s)$  and  $x^2(s)$  in the Taylor expansion for  $\tilde{\tau}(x(s))$  and  $\tilde{\sigma}(x(s))$ , respectively, i.e.,  $\tilde{\tau}(x(s)) = \tilde{\tau}'x(s) + \tilde{\tau}(0)$  and  $\tilde{\sigma}(x(s)) = \tilde{\sigma}''/2 x^2(s) + \tilde{\sigma}'(0)x(s) + \tilde{\sigma}(0)$ .

Observe that the coefficients  $C_1$  and  $C_2$  of the  $q^n$  and  $q^{-n}$  terms, respectively, are fixed by the functions  $\sigma$  and  $\tau$  in (1), and so is the product  $C_1 C_2$ . In what follows we denote by  $L_q$  the value of  $C_1 C_2 = ((\tilde{\sigma}''/k_q)^2 - (\tilde{\tau}')^2)/4k_q^2$ .

The sequence  $\{\lambda_n(q)\}$  satisfies the following three-term recurrence relation (TTRR):

$$\lambda_{n+2}(q) - (q + q^{-1})\lambda_{n+1}(q) + \lambda_n(q) = \frac{1}{2}(\tilde{\tau}'k_q^2 - \tilde{\sigma}''[2]_q) = C. \quad (4)$$

Conversely, if  $\{\lambda_n(q)\}$  satisfy the TTRR (4), then it has the form  $\lambda_n(q) = C'_1 q^n + C'_2 q^{-n} + C'_3$ . Obviously, having used the initial conditions  $\lambda_0(q) = 0$  and  $\lambda_1(q) = -\tilde{\tau}'$ , one recovers the expression (3).

It is well known [3] that under certain conditions the polynomial solutions of (1) are orthogonal. For example, if  $\sigma(s)\rho(s)x^k(s - \frac{1}{2})|_{s=a,b} = 0$ , for all  $k = 0, 1, 2, \dots$ , then the polynomial solutions possess a discrete orthogonality property

$$\sum_{s=a}^{b-1} P_n(s; q) P_m(s; q) \rho(s) \nabla x_1(s) = d_n^2(q) \delta_{n,m}, \quad (5)$$

where the weight function  $\rho(s)$  is a solution of the Pearson-type difference equation [3]

$$\frac{\Delta}{\Delta x(s - \frac{1}{2})} [\sigma(s)\rho(s)] = \tau(s)\rho(s) \quad \text{or} \quad \sigma(s+1)\rho(s+1) = \sigma(-s - \mu)\rho(s). \quad (6)$$

If the lattice  $x(s)$  is a  $q$ -linear lattice, i.e.,  $x(s) = cq^{\pm s} + c_3$ , then the  $\sigma(-s - \mu)$  in (6) should be substituted by  $\sigma(s) + \tau(s)\Delta x(s - 1/2)$ . A more detailed information on orthogonal polynomials on the non-uniform lattices can be found in [3, 5–8].

In [1] it has been shown that one can factorize the Nikiforov–Uvarov equation (1) with the aid of raising and lowering operators, which can be constructed for solutions of this equation. In this paper we wish to go one step further by studying the dynamical symmetry algebra for the hypergeometric-type difference equation (1) on the non-uniform lattices (2). Our approach is essentially based on the simple observation, formulated in [9]. In order to factorize an arbitrary difference equation, one should express it explicitly in terms of the shift (or displacement) operators  $\exp(a \frac{d}{ds})$ , which are defined as  $\exp(a \frac{d}{ds})f(s) = f(s+a)$ ,  $a$  is some constant. For example, in the case of equation (1) this corresponds to the substitutions  $\Delta = \exp(\frac{d}{ds}) - 1$  and  $\nabla = 1 - \exp(-\frac{d}{ds})$ . This procedure converts a difference equation into an eigenvalue problem for a difference operator, represented by a linear combination of some shift operators (with coefficients, which depend polynomially on the variable  $s$ ). Since each term of this linear combination is readily factorizable (because  $\exp(\alpha + \beta)A = \exp \alpha A \exp \beta A$ , for an

arbitrary operator  $A$ ), the factorization of the whole linear combination, which represents the initial difference equation, becomes straightforward.

Inspired by the appearance of Macfarlane's [10] and Biedenharn's [11] important constructions of  $q$ -analogues of the quantum harmonic oscillator, this technique of factorization of difference equations was later employed in a number of publications [12–16] in order to study group theoretic properties of the various well-known families of orthogonal polynomials, which can be viewed as  $q$ -extensions of the classical Hermite polynomials. So our purpose here is to formulate a unified approach to deriving all of these results, which correspond to the  $q$ -linear spectrum.

An important aspect to observe at this point is that we shall mainly (except for the examples in subsection 4.2) confine our attention to those families of  $q$ -polynomials, which satisfy discrete orthogonality relation of the type (5). The explanation of such preference is that the factorization of difference equations for instances of  $q$ -polynomials with continuous orthogonality property has been already thoroughly studied in [12–16]. Observe also that our approach still remains valid in the limit as  $q \rightarrow 1$ ; so classical counterparts of  $q$ -polynomials, which will be discussed in this paper, are in fact incorporated as appropriate limit cases. But the reader who desires to know more about the factorization in the cases of classical orthogonal polynomials (such as the Kravchuk, Charlier, Meixner, Meixner–Pollaczek, and Hahn) may be referred to [17, 18] and references therein.

The paper is organized as follows. In section 2 we associate with each family of  $q$ -polynomials a ' $q$ -Hamiltonian'  $\mathfrak{H}(s; q)$  (via the second-order difference equation) and construct two difference operators  $a(s; q)$  and  $b(s; q)$ , which factorize the operator  $\mathfrak{H}(s; q)$ . Our main results are given in section 3: they are formulated in theorems 3.4—which gives a *simple* necessary and sufficient condition that the  $q$ -Hamiltonian  $\mathfrak{H}(s; q)$  admits the factorization in terms of the operators  $a(s; q)$  and  $b(s; q)$ , which satisfy the relation  $a(s; q)b(s; q) - q^\gamma b(s; q)a(s; q) = I$  for some  $\gamma$ , and 3.5—stating that the eigenvalues of the difference equation (1) in this case should be of the form  $\lambda_n(q) = C_1 q^n + C_3$  or  $\lambda_n(q) = C_2 q^{-n} + C_3$ . In section 4 several relevant examples of particular  $q$ -families of orthogonal polynomials are illustrated.

## 2. Factorization operators

Let us introduce a set of functions  $\Phi_n$

$$\Phi_n(s; q) = d_n^{-1} A(s) \sqrt{\rho(s)} P_n(s; q), \quad (7)$$

where  $d_n$  is the norm of the  $q$ -polynomials  $P_n(s; q)$ ,  $\rho(s)$  is the solution of the Pearson equation (6), and  $A(s)$  is an arbitrary continuous function,  $A(s) \neq 0$  in the interval  $(a, b)$  of orthogonality of  $P_n$ . If the polynomials  $P_n(s; q)$  possess the discrete orthogonality property (5), then the functions  $\Phi_n(s; q)$  satisfy

$$\langle \Phi_n(s; q) \Phi_m(s; q) \rangle = \sum_{s=a}^{b-1} \Phi_n(s; q) \Phi_m(s; q) \frac{\nabla x_1(s)}{A^2(s)} = \delta_{n,m}. \quad (8)$$

Note that if  $A(s) = \sqrt{\nabla x_1(s)}$ , then the set  $(\Phi_n)_n$  is an orthonormal set. Obviously, in the case of a continuous orthogonality (as for the Askey–Wilson polynomials) one needs to change the sum in (8) by a Riemann integral [3, 5].

Next, we define the  $q$ -Hamiltonian  $\mathfrak{H}(s; q)$  of the form

$$\mathfrak{H}(s; q) := \frac{1}{\nabla x_1(s)} A(s) H(s; q) \frac{1}{A(s)}, \quad (9)$$

where

$$H(s; q) := -\frac{\sqrt{\sigma(-s-\mu+1)\sigma(s)}}{\nabla x(s)} e^{-\partial_s} - \frac{\sqrt{\sigma(-s-\mu)\sigma(s+1)}}{\Delta x(s)} e^{\partial_s} + \left( \frac{\sigma(-s-\mu)}{\Delta x(s)} + \frac{\sigma(s)}{\nabla x(s)} \right) I, \quad (10)$$

$e^{\alpha\partial_s} f(s) = f(s + \alpha)$  for all  $\alpha \in \mathbb{C}$  and  $I$  is the identity operator. If we now use the identity  $\nabla = \Delta - \nabla\Delta$  and equation (1), we find that

$$\mathfrak{H}(s; q)\Phi_n(s; q) = \lambda_n\Phi_n(s; q), \quad (11)$$

i.e., the functions  $\Phi_n(s; q)$ , defined in (7), are the eigenfunctions of the associated operator  $\mathfrak{H}(s; q)$ .

Our first step is to find two operators  $a(s; q)$  and  $b(s; q)$  such that the Hamiltonian  $\mathfrak{H}(s; q) = b(s; q)a(s; q)$ , i.e., the operators  $a(s; q)$  and  $b(s; q)$  factorize the  $q$ -Hamiltonian  $\mathfrak{H}(s; q)$ . But before exhibiting their explicit form let us point out that if there exists a pair of such operators, then there are infinitely many of them. Indeed, let  $a(s; q)$  and  $b(s; q)$  be such operators that  $\mathfrak{H}(s; q) = b(s; q)a(s; q)$  and let  $U(s; q)$  be an arbitrary unitary operator, i.e.,  $U^\dagger(s; q)U(s; q) = I$ . Then the operators

$$\tilde{a}(s; q) := U(s; q)a(s; q), \quad \tilde{b}(s; q) := b(s; q)U^\dagger(s; q),$$

also factorize  $\mathfrak{H}(s; q)$  for

$$\begin{aligned} \tilde{b}(s; q)\tilde{a}(s; q) &= b(s; q)U^\dagger(s; q)U(s; q)a(s; q) \\ &= b(s; q)Ia(s; q) = b(s; q)a(s; q) = \mathfrak{H}(s; q). \end{aligned}$$

This arbitrariness in picking up a particular unitary operator  $U(s)$  is essential because it enables one to construct a closed algebra, which contains a Hamiltonian  $\mathfrak{H}(s; q)$  itself. An explicit form of the spectrum of this Hamiltonian may then be found by purely algebraic arguments from knowledge of representations of this algebra (which is therefore referred to as a *dynamical algebra*).

If one applies the standard factorization procedure to equation (1), then the following difference operators emerge.

**Definition 2.1.** Let  $\alpha$  be a real number and  $A(s)$  an arbitrary continuous non-vanishing function in  $(a, b)$ . We define a family of  $\alpha$ -down and  $\alpha$ -up operators by

$$\begin{aligned} \alpha_\alpha^\downarrow(s; q) &:= \frac{A(s)}{\sqrt{\nabla x_1(s)}} e^{-\alpha\partial_s} \left( e^{\partial_s} \sqrt{\frac{\sigma(s)}{\nabla x(s)}} - \sqrt{\frac{\sigma(-s-\mu)}{\Delta x(s)}} \right) \frac{1}{A(s)}, \\ \alpha_\alpha^\uparrow(s; q) &:= \frac{1}{\nabla x_1(s)} A(s) \left( \sqrt{\frac{\sigma(s)}{\nabla x(s)}} e^{-\partial_s} - \sqrt{\frac{\sigma(-s-\mu)}{\Delta x(s)}} \right) e^{\alpha\partial_s} \frac{\sqrt{\nabla x_1(s)}}{A(s)}, \end{aligned} \quad (12)$$

respectively.

A straightforward calculation (by using the simple identity  $e^{\partial_s}\nabla = \Delta$ ) shows that for all  $\alpha \in \mathbb{R}$

$$\mathfrak{H}(s; q) = \alpha_\alpha^\uparrow(s; q)\alpha_\alpha^\downarrow(s; q), \quad (13)$$

i.e., the operators  $\alpha_\alpha^\downarrow(s; q)$  and  $\alpha_\alpha^\uparrow(s; q)$  factorize the Hamiltonian, defined in (9). i.e., we have the following:

**Theorem 2.2.** Given a  $q$ -Hamiltonian (9)  $\mathfrak{H}(s; q)$ , then the operators  $\alpha_\alpha^\downarrow(s; q)$  and  $\alpha_\alpha^\uparrow(s; q)$  defined in (12) are such that for all  $\alpha \in \mathbb{C}$ ,  $\mathfrak{H}(s; q) = \alpha_\alpha^\uparrow(s; q)\alpha_\alpha^\downarrow(s; q)$ .

Our next step is to find a dynamical algebra, associated with the Hamiltonian  $\mathfrak{H}(s; q)$ . To this end we will need the following definition.

**Definition 2.3.** A function  $f(z)$  is said to be a linear-type function of  $z$ , if there exist two functions  $F$  and  $G$ , such that for all  $z, \zeta \in \mathbb{C}$ , this function  $f(z)$  can be represented as

$$f(z + \zeta) = F(\zeta)f(z) + G(\zeta).$$

A particular case of the linear-type functions are the  $q$ -linear functions, i.e., the functions of the form  $f(z) = Aq^z + B$ . For these functions  $f(z + \zeta) = F(\zeta)f(z) + G(\zeta)$ , where  $F(\zeta) = q^\zeta$  and  $G(\zeta) = B(1 - q^\zeta)$ .

**Remark 2.4.** If we use the expression in (3) for the eigenvalues  $\lambda_n$ , then it is straightforward to see that  $\lambda_n$  is a  $q$ -linear function of  $n$  if and only if  $\tilde{\sigma}'' = \pm k_q \tilde{\tau}'$ . Moreover, in this case we have

$$\tilde{\sigma}'' = k_q \tilde{\tau}' \Rightarrow \lambda_n(q, +) = \frac{\tilde{\tau}'}{1 - q}(q^n - 1),$$

and

$$\tilde{\sigma}'' = -k_q \tilde{\tau}' \Rightarrow \lambda_n(q, -) = \frac{\tilde{\tau}'}{1 - q^{-1}}(q^{-n} - 1). \quad (14)$$

Note that  $\lambda_n(q, -) = \lambda_n(q^{-1}, +)$ , i.e., the second case can be obtained from the first one just by changing  $q$  to  $q^{-1}$ .

**Proposition 2.5.** The function  $\lambda_n$  is a  $q$ -linear function of  $n$  if and only if it satisfies  $\lambda_{n+1} = q\lambda_n + C$ .

**Proof.** A straightforward computation shows that if  $\lambda_n$  is a  $q$ -linear function of  $n$ , then it satisfies the recurrence formula  $\lambda_{n+1} = q\lambda_n + C$ , where  $C$  is a constant (in this case  $C = \lambda_1$ ). But the general solution of the difference equation  $\lambda_{n+1} = q\lambda_n + C$  is  $\lambda_n = Aq^n + D$ , where  $A$  and  $D$  are, in general, non-vanishing constants.  $\square$

**Remark 2.6.** Note that if  $\lambda_n$  is a  $q$ -linear function of  $n$ , then  $\lambda_n$  satisfies the recurrence relation  $\lambda_{n+\gamma} - q^\gamma \lambda_n = C$  for any numbers  $\gamma$  and  $C$ .

Finally, we have the following straightforward lemma:

**Lemma 2.7.** Let  $x(s)$  be a  $q$ -linear function of  $s$ , and  $\lambda_n$  be the eigenvalue of the difference equation of hypergeometric type (1). Then  $\lambda_n$  is a  $q$ -linear function of  $n$  if and only if  $\Delta^{(2)}(\sigma(s)) = 0$  and  $q^{-1}$ -linear function of  $n$  if and only if  $\Delta^{(2)}(\sigma(-s - \mu)) = 0$ , where  $\Delta^{(2)}$  is the operator  $\Delta^{(2)} = \frac{\Delta}{\Delta x_1(s)} \frac{\Delta}{\Delta x(s)}$ .

**Proof.** It follows from equation (14) and the fact that  $\Delta^{(2)}(\sigma(s)) = \frac{[2]_q}{2}(\tilde{\sigma}'' - \tilde{\tau}'k_q)$  and  $\Delta^{(2)}(\sigma(-s - \mu)) = \frac{[2]_q}{2}(\tilde{\sigma}'' + \tilde{\tau}'k_q)$ .  $\square$

### 3. Dynamical algebra

We begin this section with the following definition.

**Definition 3.1.** Let  $\zeta$  be a real number, and let  $a(s; q)$  and  $b(s; q)$  be two operators. We define the  $\zeta$ -commutator of  $a$  and  $b$  as

$$[a(s; q), b(s; q)]_\zeta = a(s; q)b(s; q) - \zeta b(s; q)a(s; q).$$

**Proposition 3.2.** Let  $\mathfrak{H}(s; q)$  be an operator, such that there exist two operators  $a(s; q)$  and  $b(s; q)$  and two real numbers  $\zeta$  and  $\Lambda$ , such that  $\mathfrak{H}(s; q) = b(s) a(s; q)$ , and  $[a(s; q), b(s; q)]_{\zeta} = \Lambda$ . Then, if  $\Phi(s; q)$  is an eigenvector of the Hamiltonian  $\mathfrak{H}(s; q)$ , associated with the eigenvalue  $\lambda$ , we have

1.  $\mathfrak{H}(s; q)\{a(s; q)\Phi(s; q)\} = \zeta^{-1}(\lambda - \Lambda)\{a(s; q)\Phi(s; q)\}$ , i.e.,  $a(s; q)\Phi(s; q)$  is the eigenvector of  $\mathfrak{H}(s; q)$ , associated with the eigenvalue  $\zeta^{-1}(\lambda - \Lambda)$ ,
2.  $\mathfrak{H}(s; q)\{b(s; q)\Phi(s; q)\} = (\Lambda + \zeta\lambda)\{b(s; q)\Phi(s; q)\}$ , i.e.,  $b(s; q)\Phi(s; q)$  is the eigenvector of  $\mathfrak{H}(s; q)$ , associated with the eigenvalue  $\Lambda + \zeta\lambda$ .

**Proof.** In the first case, since  $\mathfrak{H}(s; q)\Phi(s; q) = \lambda\Phi(s; q)$ ,

$$\begin{aligned}\mathfrak{H}(s; q)\{a(s; q)\Phi(s; q)\} &= b(s; q)a(s; q)\{a(s; q)\Phi(s; q)\} \\ &= \zeta^{-1}(a(s; q)b(s; q) - \Lambda)\{a(s; q)\Phi(s; q)\} \\ &= \zeta^{-1}(\lambda - \Lambda)\{a(s; q)\Phi(s; q)\}.\end{aligned}$$

By the same token, in the second case

$$\begin{aligned}\mathfrak{H}(s; q)\{b(s; q)\Phi(s; q)\} &= b(s; q)a(s; q)b(s; q)\Phi(s; q) = b(s; q)(\Lambda + \zeta\lambda)\Phi(s; q) \\ &= (\Lambda + \zeta\lambda)\{b(s; q)\Phi(s; q)\}.\end{aligned}\quad \square$$

In the same way one can prove that

$$a(s; q)b(s; q)\Phi(s; q) = (\Lambda + \zeta\lambda)\Phi(s; q). \quad (15)$$

Moreover, if  $\Phi(s; q)$  is an eigenvector of the Hamiltonian  $\mathfrak{H}(s; q)$  (or of the operator  $a(s; q)b(s; q)$ ), then  $a^k(s; q)\Phi(s; q)$  and  $b^k(s; q)\Phi(s; q)$  are, in general, also eigenvectors.

**Remark 3.3.** Obviously, the condition  $[a(s; q), b(s; q)]_{\zeta} = I$  can be changed to  $[a(s; q), b(s; q)]_{\zeta} = \Lambda$ , where  $\Lambda$  is an arbitrary nonzero constant. In fact, if the operators  $a(s; q)$  and  $b(s; q)$  satisfy the  $q$ -commutation relation  $[a(s; q), b(s; q)]_{\zeta} = \Lambda$ , then the operators  $\mathfrak{a}(s; q) = \Lambda^{-1/2}a(s; q)$  and  $\mathfrak{b}(s; q) = \Lambda^{-1/2}b(s; q)$  satisfy  $[\mathfrak{a}(s; q), \mathfrak{b}(s; q)]_{\zeta} = I$  and  $\mathfrak{H}(s; q) = \Lambda b(s; q)a(s; q)$ .

Proposition 3.2 thus refers to the case of a system, described by a Hamiltonian  $\mathfrak{H}(s; q)$ , which admits the factorization (13) in terms of the operators  $a(s; q)$  and  $b(s; q)$ , satisfying the  $q$ -commutation relation  $[a(s; q), b(s; q)]_{\zeta} = I$ . Moreover, it tells us how to construct a dynamical symmetry algebra for such a case in a direct fashion [19]. Indeed, let us assume that  $[a(s; q), b(s; q)]_{\zeta} = I$ ,  $\zeta = q^2$  (or  $q^{-2}$ ) and  $b(s; q) = a^{\dagger}(s; q)$ . Then one can rewrite the  $q^2$ -commutator  $a(s; q)a^{\dagger}(s; q) - q^2a^{\dagger}(s; q)a(s; q) = I$  in the following form

$$\begin{aligned}[a(s; q), a^{\dagger}(s; q)] &:= a(s; q)a^{\dagger}(s; q) - a^{\dagger}(s; q)a(s; q) \\ &= I - (1 - q^2)a^{\dagger}(s; q)a(s; q) := q^{2N(s)},\end{aligned}$$

where, by definition, the operator  $N(s)$  is equal to  $N(s) = \ln[I - (1 - q^2)a^{\dagger}(s; q)a(s; q)] / \ln q^2$ . From this definition of  $N(s)$  it follows that

$$[N(s), a(s; q)] = -a(s; q), \quad [N(s), a^{\dagger}(s; q)] = a^{\dagger}(s; q), \quad (16)$$

i.e.,  $N(s)$  is the number operator. The next (and final) step is to introduce a new set of the operators

$$b(s; q) := q^{-N(s)/2}a(s; q), \quad b^{\dagger}(s; q) := a^{\dagger}(s; q)q^{-N(s)/2},$$

which satisfy the following commutation relation:

$$b(s; q)b^{\dagger}(s; q) - qb^{\dagger}(s; q)b(s; q) = q^{-N(s)},$$

readily derived with the aid of (16). The operators  $b(s; q)$ ,  $b^\dagger(s; q)$ , and  $N(s)$  directly lead to the dynamical algebra  $su_q(1, 1)$  with the generators

$$\begin{aligned} K_0(s) &= \frac{1}{2}[N(s) + 1/2], & K_+(s) &= \beta(b^\dagger(s; q))^2, \\ K_-(s) &= \beta b^2(s; q), & \beta^{-1} &= q + q^{-1}. \end{aligned}$$

It is straightforward to verify that the generators thus defined satisfy the standard commutation relations

$$[K_0(s), K_\pm(s)] = \pm K_\pm(s), \quad [K_-(s), K_+(s)] = [2K_0(s)]_{q^2},$$

of the algebra  $su_q(1, 1)$  (see e.g. [20]).

Thus in the case when the operators  $a(s; q)$  and  $b(s; q)$ , which factorize the Hamiltonian  $\mathfrak{H}(s; q)$ , satisfy the  $q$ -commutation relation  $[a(s; q), b(s; q)]_{\zeta^2} = I$  and  $b(s; q) = a^\dagger(s; q)$ , the appropriate dynamical symmetry algebra is  $su_\zeta(1, 1)$ . So the question arises: what are the conditions for insuring that such  $q$ -commutator takes place? In other words, we have the following:

**Problem 1.** *To find two operators  $a(s; q)$  and  $b(s; q)$  and a constant  $\zeta$  such that the Hamiltonian  $\mathfrak{H}(s; q) = b(s; q)a(s; q)$  and  $[a(s; q), b(s; q)]_\zeta = I$ .*

For the first part we already have the answer (see theorem 2.2). The solution of the second one is formulated in the following two theorems.

**Theorem 3.4.** *Let  $\mathfrak{H}(s; q)$  be the following difference operator ( $q$ -Hamiltonian):*

$$\mathfrak{H}(s; q) = \frac{1}{\nabla x_1(s)} A(s) H(s; q) \frac{1}{A(s)}. \quad (17)$$

*The operators  $b(s; q) = \alpha_\alpha^\dagger(s; q)$  and  $a(s; q) = \alpha_\alpha^\downarrow(s; q)$  given in (12) factorize the Hamiltonian  $\mathfrak{H}(s; q)$  (17) and satisfy the commutation relation  $[a(s; q), b(s; q)]_\zeta = \Lambda$  for a certain real number  $\zeta$  if and only if the following two conditions hold:*

$$\frac{\nabla x(s)}{\nabla x_1(s - \alpha)} \sqrt{\frac{\nabla x_1(s - 1) \nabla x_1(s)}{\nabla x(s - \alpha) \Delta x(s - \alpha)}} \sqrt{\frac{\sigma(s - \alpha) \sigma(-s - \mu + \alpha)}{\sigma(s) \sigma(-s - \mu + 1)}} = \zeta, \quad (18)$$

and

$$\frac{1}{\Delta x(s - \alpha)} \left( \frac{\sigma(s - \alpha + 1)}{\nabla x_1(s - \alpha + 1)} + \frac{\sigma(-s - \mu + \alpha)}{\nabla x_1(s - \alpha)} \right) - \zeta \frac{1}{\nabla x_1(s)} \left( \frac{\sigma(s)}{\nabla x(s)} + \frac{\sigma(-s - \mu)}{\Delta x(s)} \right) = \Lambda. \quad (19)$$

**Proof.** Taking the expression of the operators  $\alpha_\alpha^\dagger(s)$  and  $\alpha_\alpha^\downarrow(s)$ , a straightforward calculus shows that  $\alpha_\alpha^\downarrow(s) \alpha_\alpha^\dagger(s) = A_1(s) e^{\partial_s} + A_2(s) e^{-\partial_s} + A_3(s) I$ , where

$$\begin{aligned} A_1(s) &= -\sqrt{\frac{\nabla x_1(s + 1)}{\nabla x_1(s)}} \frac{A(s)}{A(s + 1)} \sqrt{\frac{\sigma(s + 1 - \alpha) \sigma(-s - \mu - 1 + \alpha)}{\Delta x(s - \alpha) \Delta x(s + 1 - \alpha)}} \frac{1}{\nabla x_1(s + 1 - \alpha)}, \\ A_2(s) &= -\sqrt{\frac{\nabla x_1(s - 1)}{\nabla x_1(s)}} \frac{A(s)}{A(s - 1)} \sqrt{\frac{\sigma(s - \alpha) \sigma(-s - \mu + \alpha)}{\Delta x(s - 1 - \alpha) \Delta x(s - \alpha)}} \frac{1}{\nabla x_1(s - \alpha)}, \\ A_3(s) &= \frac{1}{\Delta x(s - \alpha)} \left[ \frac{\sigma(s + 1 - \alpha)}{\nabla x_1(s + 1 - \alpha)} + \frac{\sigma(-s - \mu + \alpha)}{\nabla x_1(s - \alpha)} \right]. \end{aligned} \quad (20)$$



In the same way, using (10) and (9) we have  $a_\alpha^\uparrow(s)a_\alpha^\downarrow(s) = \mathfrak{H}(s; q) = B_1(s)e^{\partial_s} + B_2(s)e^{-\partial_s} + B_3(s)I$ , where

$$\begin{aligned} B_1(s) &= -\frac{1}{\nabla x_1(s)} \frac{A(s)}{A(s+1)} \frac{\sqrt{\sigma(-s-\mu)\sigma(s+1)}}{\nabla x(s+1)}, \\ B_2(s) &= -\frac{1}{\nabla x_1(s)} \frac{A(s)}{A(s-1)} \frac{\sqrt{\sigma(-s-\mu+1)\sigma(s)}}{\nabla x(s)}, \\ B_3(s) &= \frac{1}{\nabla x_1(s)} \left[ \frac{\sigma(s)}{\nabla x(s)} + \frac{\sigma(-s-\mu)}{\Delta x(s)} \right]. \end{aligned} \quad (21)$$

Consequently,

$$[a_\alpha^\downarrow(s), a_\alpha^\uparrow(s)]_\zeta = (A_1(s) - \zeta B_1(s))e^{\partial_s} + (A_2(s) - \zeta B_2(s))e^{-\partial_s} + (A_3(s) - \zeta B_3(s))I. \quad (22)$$

To eliminate the two terms in the right-hand side of (22), which are proportional to the difference operators  $\exp(\pm\partial_s)$ , one must require that

$$A_1(s) - \zeta B_1(s) = 0, \quad A_2(s) - \zeta B_2(s) = 0. \quad (23)$$

Offhand, it is not evident that one can satisfy both of the relations (23), which only involve the same constant  $\zeta$ . But it is straightforward to verify from (20) and (21) that

$$A_1(s)B_2(s+1) = A_2(s+1)B_1(s),$$

or, equivalently,

$$\frac{A_1(s)}{B_1(s)} = \frac{A_2(s+1)}{B_2(s+1)}.$$

Hence, the requirement that  $A_1(s) = \zeta B_1(s)$  entails the relation  $A_2(s) = \zeta B_2(s)$ , and vice versa. From (22) it is now evident that the commutator  $[a_\alpha^\uparrow(s), a_\alpha^\downarrow(s)]_\zeta$  is a constant if (23) holds and the factor  $A_3(s) - \zeta B_3(s)$  is a constant. Thus, the required conditions (18) and (19) immediately follow.  $\square$

**Theorem 3.5.** *Let  $(\Phi_n)_n$  be the eigenfunctions of  $\mathfrak{H}(s; q)$  corresponding to the eigenvalues  $(\lambda_n)_n$  and suppose that problem 1 has a solution for  $\Lambda \neq 0$ . Then, the eigenvalues  $\lambda_n$  of the difference equation (11) are  $q$ -linear or  $q^{-1}$ -linear functions of  $n$ , i.e.,  $\lambda_n = C_1q^n + C_3$  or  $\lambda_n = C_2q^{-n} + C_3$ , respectively.*

**Proof.**<sup>5</sup> In the following we use the notation  $\zeta = q^\gamma$ . Suppose that problem 1 has a solution with  $\Lambda \neq 0$ , and  $\lambda_n$  is not a  $q$ -linear (respectively,  $q^{-1}$ ) function of  $n$ . From proposition 3.2 we know that  $a_\alpha^\uparrow(s; q)\Phi_n(s; q)$  is an eigenvector of  $\mathfrak{H}(s; q)$  corresponding to the eigenvalue  $\Lambda + q^\gamma\lambda_n$ . If we denote by  $\Phi_{m(n);q}$  such eigenvector where  $m(n)$  is a function of  $n$ , then we have  $\Lambda + q^\gamma\lambda_n = \lambda_{m(n)}$ . Then using (3) we get

$$\lambda_{m(n)} = C_1q^{m(n)} + C_2q^{-m(n)} + C_3, \quad C_1C_2 = L_q. \quad (24)$$

On the other hand,

$$\lambda_{m(n)} = \Lambda + q^\gamma\lambda_n = C_1q^\gamma q^n + C_2q^\gamma q^{-n} + q^\gamma C_3 + \Lambda = C'_1q^n + C'_2q^{-n} + C'_3. \quad (25)$$

But here, since  $\lambda_{m(n)}$  is an eigenvalue of (1), again we have the condition  $C'_1C'_2 = L_q$ , thus  $C_1C_2 = L_q = C'_1C'_2 = C_1C_2q^{2\gamma}$  so  $q^{2\gamma} = 1$ , i.e.,  $\gamma = 0$ , or  $C_1C_2 = 0$ . In the first case, equating (24) and (25), we have that  $C'_3 = C_3q^\gamma + \Lambda = C_3$ , i.e.,  $\Lambda = 0$  that is a contradiction. Thus  $C_1C_2 = 0$  from where the result easily follows.  $\square$

<sup>5</sup> For an alternative proof in the case  $\alpha = 0$  see the appendix.

It is worth noting that the  $q$ -linearity of the eigenvalues of  $\mathfrak{H}(s; q)$  is only the necessary condition, i.e., it is not sufficient. So there are cases when the eigenvalues  $\lambda_n(q)$  are  $q$ -linear (for instance, those which correspond to the  $q$ -Meixner, the  $q$ -Charlier and the  $q$ -Laguerre polynomials with  $a \neq q^{-1/2}$ ), but the corresponding  $q$ -Hamiltonians  $\mathfrak{H}(s; q)$  do not admit the factorization in terms of  $q$ -commuting operators. This just reflects the fact that an appropriate dynamical algebra is not  $su_q(1, 1)$ , and one has to consider a more complicated quadratic algebra  $AW(3)$  [21]. The problem of finding an explicit connection between the generators of the algebra  $AW(3)$  and the operators  $a(s; q)$  and  $b(s; q)$ , which factorize the  $q$ -Hamiltonians for these cases, will be addressed in a separate publication.

**Remark 3.6.** A special important case of the nonlinear lattice is when  $x(s) = \frac{1}{2}(q^s + q^{-s})$ . In this case if we put  $\alpha = \frac{1}{2}$ , then the conditions (18) and (19) of theorem 3.4 becomes

$$\sqrt{\frac{\sigma(s - \frac{1}{2})\sigma(-s + \frac{1}{2})}{\sigma(s)\sigma(-s + 1)}} = \varsigma,$$

and

$$\frac{1}{\nabla x_1(s)} \left( \frac{\sigma(s + \frac{1}{2})}{\Delta x(s)} + \frac{\sigma(-s + \frac{1}{2})}{\nabla x(s)} \right) - \varsigma \frac{1}{\nabla x_1(s)} \left( \frac{\sigma(s)}{\nabla x(s)} + \frac{\sigma(-s)}{\Delta x(s)} \right) = \Lambda,$$

respectively. Moreover, if we put  $A(s) = 1$  then,  $\mathfrak{H}(s; q) = (\nabla x_1(s))^{-1} H(s; q)$  and the  $\alpha$ -operators simplify

$$\begin{aligned} a_{1/2}^\downarrow(s) &= \frac{1}{\nabla x_1(s)} (e^{\frac{1}{2}\partial_s} \sqrt{\sigma(s)} - e^{-\frac{1}{2}\partial_s} \sqrt{\sigma(-s)}), \\ a_{1/2}^\uparrow(s) &= \frac{1}{\nabla x_1(s)} (\sqrt{\sigma(s)} e^{-\frac{1}{2}\partial_s} - \sqrt{\sigma(-s)} e^{\frac{1}{2}\partial_s}). \end{aligned}$$

Now we can formulate the

**Problem 2.** To find two operators  $a(s; q)$  and  $b(s; q)$  and a constant  $\varsigma$  such that the Hamiltonian  $\mathfrak{H}(s; q) = b(s; q)a(s; q)$  and  $[a(s; q), b(s; q)]_\varsigma = I$  and such that  $a(s; q)$  and  $b(s; q)$  are the lowering and raising operators, i.e.,

$$a(s; q)\Phi_n(s; q) = D_n\Phi_{n-1}(s; q) \quad \text{and} \quad b(s; q)\Phi_n(s; q) = U_n\Phi_{n+1}(s; q). \quad (26)$$

Again, without loss of generality, we will change the condition  $[a(s; q), b(s; q)]_\varsigma = I$  into

$$[a(s; q), b(s; q)]_\varsigma = \Lambda \quad \text{and chose} \quad \Lambda = \lambda_1.$$

Also the operators  $b(s; q) = a_\alpha^\uparrow(s; q)$  and  $a(s; q) = a_\alpha^\downarrow(s; q)$ , given in (12), provide the factorization of  $\mathfrak{H}(s; q)$ .

If we now apply  $b(s; q)$  to the first equation of (26) and use the second one as well as (11), we find that  $\lambda_n = D_n U_{n-1}$ . On the other hand, applying  $a(s; q)$  to the second equation in (26) and using the first one as well as (15) we obtain  $\lambda_1 + \varsigma\lambda_n = U_n D_{n+1} = \lambda_{n+1}$ . Thus, using proposition 2.5 we conclude that  $\lambda_n$  should be a  $\varsigma$ -linear function, i.e.,  $\lambda_n$  has the form  $\lambda_n = A\varsigma^n + D$ , where  $A$  and  $D$  are non-vanishing constants. Moreover, using the recurrence  $\lambda_1 + \varsigma\lambda_n = \lambda_{n+1}$  and proposition 3.2 we obtain that

$$\begin{aligned} \mathfrak{H}(s; q)\{a(s; q)\Phi_n(s; q)\} &= \lambda_{n-1}\{a(s; q)\Phi_n(s; q)\}, \\ \mathfrak{H}(s; q)\{b(s; q)\Phi_n(s; q)\} &= \lambda_{n+1}\{b(s; q)\Phi_n(s; q)\}, \end{aligned}$$

i.e.,  $a(s; q)\Phi_n(s; q)$  is the eigenvector associated with the eigenvalue  $\lambda_{n-1}$  and  $b(s; q)\Phi_n(s; q)$  is the eigenvector associated with the eigenvalue  $\lambda_{n+1}$ , so they really are the lowering and raising operators associated with the Hamiltonian  $\mathfrak{H}(s; q)$ .

The next important question to be considered is when the  $\alpha$ -operators are mutually adjoint? Obviously the answer to this question depends on how the scalar product is defined. As an example, let us consider the discrete orthogonality case (8) and compute  $\langle \mathfrak{a}_\alpha^\downarrow \Phi_{n+1}, \Phi_k \rangle$ :

$$\begin{aligned} & \langle \mathfrak{a}_\alpha^\downarrow(s; q)\Phi_{n+1}(s; q), \Phi_k(s; q) \rangle \\ &= \sum_{s=a}^{b-1} \frac{\sqrt{\nabla x_1(s)\rho(s)}}{d_n d_{n+1}} \left[ e^{-\alpha \partial_s} \left( \sqrt{\frac{\sigma(s+1)\rho(s+1)}{\nabla x(s+1)}} P_{n+1}(s+1; q) \right. \right. \\ & \quad \left. \left. - \sqrt{\frac{\sigma(-s-\mu)\rho(s)}{\Delta x(s)}} P_{n+1}(s; q) \right) \right] P_k(s; q) \\ &= \sum_{s=a}^{b-1} \frac{\sqrt{\nabla x_1(s)\rho(s)}}{d_n d_{n+1}} \left[ e^{-\alpha \partial_s} \left( \sqrt{\frac{\sigma(-s-\mu)\rho(s)}{\Delta x(s)}} \Delta P_{n+1}(s; q) \right) \right] P_k(s; q) \\ &= \sum_{s=a}^{b-1} \frac{\sqrt{\nabla x_1(s)\rho(s)}}{d_n d_{n+1}} \sqrt{\frac{\sigma(-s-\mu+\alpha)\rho(s-\alpha)}{\Delta x(s-\alpha)}} \Delta P_{n+1}(s-\alpha; q) P_k(s; q), \end{aligned}$$

where the second equality follows from the Pearson equation (6). Next we have

$$\begin{aligned} & \langle \Phi_{n+1}(s; q), \mathfrak{a}_\alpha^\uparrow(s)\Phi_k(s; q) \rangle \\ &= \underbrace{\sum_{s=a}^{b-1} \frac{\sqrt{\sigma(s)\rho(s)}}{d_n d_{n+1} \sqrt{\nabla x(s)}} [e^{-\alpha \partial_s} (e^{\alpha \partial_s} \sqrt{\nabla x_1(s)\rho(s)} P_k(s; q))] P_{n+1}(s; q)}_{S_1} \\ & \quad - \underbrace{\sum_{s=a}^{b-1} \frac{\sqrt{\sigma(-s-\mu)\rho(s)}}{d_n d_{n+1} \sqrt{\Delta x(s)}} [e^{\alpha \partial_s} (\sqrt{\nabla x_1(s)\rho(s)} P_k(s; q))] P_{n+1}(s; q)}_{S_2}. \end{aligned}$$

If we use the boundary conditions  $\sigma(s)\rho(s)|_{s=a,b} = 0$ , then the sum  $S_1$  becomes

$$\begin{aligned} S_1 &= \sum_{s=a+1}^b \frac{\sqrt{\sigma(s)\rho(s)}}{d_n d_{n+1} \sqrt{\nabla x(s)}} [e^{-\alpha \partial_s} (e^{\alpha \partial_s} \sqrt{\nabla x_1(s)\rho(s)} P_k(s; q))] P_{n+1}(s; q) \quad (s \rightarrow s+1) \\ &= \sum_{s=a}^{b-1} \frac{\sqrt{\sigma(s+1)\rho(s+1)}}{d_n d_{n+1} \sqrt{\nabla x(s+1)}} [(e^{\alpha \partial_s} \sqrt{\nabla x_1(s)\rho(s)} P_k(s; q))] P_{n+1}(s+1; q) \\ &= \sum_{s=a}^{b-1} \frac{\sqrt{\sigma(-s-\mu)\rho(s)}}{d_n d_{n+1} \sqrt{\Delta x(s)}} [e^{\alpha \partial_s} (\sqrt{\nabla x_1(s)\rho(s)} P_k(s; q))] P_{n+1}(s+1; q), \end{aligned}$$

where the last equality holds due to the Pearson equation (6). Now subtracting  $S_1 - S_2$  yields

$$\begin{aligned} & \langle \Phi_{n+1}(s; q), \mathfrak{a}_\alpha^\uparrow(s)\Phi_k(s; q) \rangle \\ &= \sum_{s=a}^{b-1} \frac{\sqrt{\sigma(-s-\mu)\rho(s)}}{d_n d_{n+1} \sqrt{\Delta x(s)}} [e^{\alpha \partial_s} (\sqrt{\nabla x_1(s)\rho(s)} P_k(s; q))] \Delta P_{n+1}(s; q) \\ &= \sum_{s=a}^{b-1} \frac{\sqrt{\sigma(-s-\mu)\rho(s)}}{d_n d_{n+1} \sqrt{\Delta x(s)}} \sqrt{\nabla x_1(s+\alpha)\rho(s+\alpha)} P_k(s+\alpha; q) \Delta P_{n+1}(s; q). \end{aligned}$$

Then, in the discrete case, a sufficient condition for the operators  $\mathfrak{a}_\alpha^\uparrow(s; q)$  and  $\mathfrak{a}_\alpha^\downarrow(s; q)$  to be mutually adjoint, i.e.,  $\langle \mathfrak{a}_\alpha^\downarrow(s) \Phi_{n+1}(s; q), \Phi_k(s; q) \rangle = \langle \Phi_{n+1}(s; q), \mathfrak{a}_\alpha^\uparrow(s) \Phi_k(s; q) \rangle$ , is that  $\alpha = 0$ . For the general case as well as for the discrete cases when  $\alpha \neq 0$ , the problem requires a more detailed study of each case. A discussion of these cases will be considered elsewhere.

#### 4. Examples

In this section we will exhibit several examples involving some well-known families of  $q$ -polynomials. Let us point out that from the whole  $q$ -Askey tableau [7] we need only to consider those cases, when  $\lambda_n$  is a  $q$ -linear or  $q^{-1}$ -linear function of  $n$  (see theorem 3.5).

##### 4.1. The linear lattice $x(s) = c_1 q^s + c_3$ : the $q$ -Hahn tableau

We start with the so-called  $q$ -Hahn tableau (see e.g. [7, 8] and references therein). Taking into account 3, the  $\alpha$ -operators are chosen in such a way that  $A(s) = \sqrt{\nabla x_1(s)}$ , so the orthonormal functions are defined by

$$\Phi_n(s; q) = d_n^{-1} \sqrt{\rho(s) \nabla x_1(s)} P_n(s; q)$$

and satisfy the discrete orthogonality relation

$$\sum_{s_i=a}^{b-1} \Phi_n(s_i; q) \Phi_m(s_i; q) = \delta_{nm}.$$

By using theorem 3.4, one can solve the factorization problem 1 for the  $q$ -Hahn tableau.

Before starting with some relevant examples, we recall that for the  $q$ -linear lattice  $x(s) = c_1 q^s + c_3$ ,

$$\mathfrak{e}_q(s) := \frac{\nabla x(s)}{\nabla x_1(s - \alpha)} \sqrt{\frac{\nabla x_1(s-1) \nabla x_1(s)}{\nabla x(s - \alpha) \Delta x(s - \alpha)}} = q^{2\alpha-1},$$

whereas if  $x(s) = c_1 q^{-s} + c_3$ , then  $\mathfrak{e}_q(s) = q^{-2\alpha+1}$ .

**Remark 4.1.** Let  $x(s) = q^s$ . Then in the case when  $\sigma(s)$  or  $\sigma(-s - \mu) = \sigma(s) + \tau(s) \nabla x_1(s)$  are constants, the operators (12) define a dynamical algebra if and only if  $\alpha = 1$  and  $\zeta = q$  or  $\alpha = 0$  and  $\zeta = q^{-1}$ , respectively. To prove this assertion it is sufficient to use the formula (18) that yields, for the first case

$$\mathfrak{e}_q(s) \sqrt{\frac{\sigma(-s - \mu + \alpha)}{\sigma(-s - \mu + 1)}} = q^{2\alpha-1} \sqrt{\frac{\sigma(-s - \mu + \alpha)}{\sigma(-s - \mu + 1)}} = \zeta.$$

Choosing  $\alpha = 1$  we have that  $\zeta = q$ . The other case is analogous. These cases constitute the most simple ones.

4.1.1. *Stieltjes–Wigert  $q$ -polynomials  $S_n(x; q)$*  The Stieltjes–Wigert functions in  $x(s) = q^s$ , i.e. the functions, associated with the Stieltjes–Wigert polynomials, are defined by

$$\Phi_n(x; q) = \frac{1}{d_n} \sqrt{(-x, -q/x; q)_\infty} {}_1\phi_1 \left( \begin{matrix} q^{-n} \\ 0 \end{matrix} \middle| q; -xq^{n+1} \right), \quad x(s) = q^s,$$

$$d_n = \frac{q^{n/2}}{(q; q)_\infty} \sqrt{\frac{(q^{n+1}; q)_\infty}{\log q^{-1}}}.$$

In this case we have chosen  $A(s) = \sqrt{\nabla x_1(s)}$ . The above functions  $\Phi_n(x; q)$  possess the following orthogonality property  $\int_0^\infty \Phi_n(x; q) \Phi_m(x; q) dx = \delta_{n,m}$ .

Since for the Stieltjes–Wigert  $q$ -polynomials  $\sigma(s) = q^{s-1}$  and  $\sigma(s) + \tau(s)\nabla x_1(s) = q^{2s}$  [7], one can define the following Hamiltonian:

$$\mathfrak{H}(s; q) = \frac{1}{(1-q)} \{ (1+q^{-s})I - q^{-(s+1)/2} e^{\partial_s} - q^{-s/2} e^{-\partial_s} \},$$

for which one has  $\mathfrak{H}(s; q)\Phi_n(q^s; q) = \frac{1-q^n}{1-q}\Phi_n(q^s; q)$ .

Next we check the conditions of theorem 3.4: the first condition (18) yields  $\zeta = q^{\alpha/2}$  and substituting it in the second one, we obtain that it holds when  $\alpha = 2$ , i.e.,  $\zeta = q$  and therefore

$$\begin{aligned} \mathfrak{a}^\downarrow(s; q) &:= \mathfrak{a}_2^\downarrow(s; q) = \frac{1}{\sqrt{1-q}} (e^{-2\partial_s} - e^{-\partial_s} q^{-s/2}), \\ \mathfrak{a}^\uparrow(s; q) &:= \mathfrak{a}_2^\uparrow(s; q) = \frac{1}{\sqrt{1-q}} (e^{2\partial_s} - q^{-s/2} e^{\partial_s}), \end{aligned}$$

$\mathfrak{H}(s; q) = \mathfrak{a}^\uparrow(s; q)\mathfrak{a}^\downarrow(s; q)$  and  $[\mathfrak{a}^\downarrow(s; q), \mathfrak{a}^\uparrow(s; q)]_q = I$ . It is not hard to verify that in this case the operators  $\mathfrak{a}^\uparrow(s; q)$  and  $\mathfrak{a}^\downarrow(s; q)$  are the lowering and raising operators for the functions  $\Phi_n(q^s; q)$  (see [7], p 117, (3.27.6) and (3.27.8)). We remind the reader that the moment problem, associated with the Stieltjes–Wigert polynomials, is indeterminate [22, 23] and therefore there are several distinct weight functions (both continuous and discrete ones), with respect to which they are orthogonal. A similar result for the case of a discrete orthogonality condition was first considered in [12].

4.1.2. *Al-Salam and Carlitz I and II  $q$ -polynomials  $U_n^{(a)}(x; q)$  and  $V_n^{(a)}(x; q)$ .* The Al-Salam–Carlitz polynomials of types I and II depend on an additional parameter  $a$  and therefore they occupy the next level in the Askey scheme (see [7 p 114]). Since these two families are interrelated,

$$V_n^{(a)}(x; q) = U_n^{(a)}(x; q^{-1}),$$

it is sufficient to consider only one of them.

Let us define the functions [7, p 114]

$$\begin{aligned} \Phi_n(s; q) &= \frac{1}{d_n} \frac{a^{s/2} q^{s^2/2}}{\sqrt{(q, aq; q)_s}} {}_2\phi_0 \left( \begin{matrix} q^{-n}, q^{-s} \\ - \end{matrix} \middle| q; \frac{q^n}{a} \right), \quad a > 0, \\ d_n &= (-1)^n (aq)^{-n/2} \sqrt{\frac{(q; q)_n}{(aq; q)_\infty}}, \end{aligned}$$

which satisfy the discrete orthogonality relation  $\sum_{k=0}^\infty \Phi_n(k; q) \Phi_m(k; q) = \delta_{n,m}$ . Since for this family  $\sigma(s) = (q^{-s} - 1)(q^{-s} - a)$  and  $\sigma(s) + \tau(s)\nabla x_1(s) = a$ , where  $x(s) = q^{-s}$ , then one derives that the difference Hamiltonian  $\mathfrak{H}(s; q)$  has the following form

$$\begin{aligned} \mathfrak{H}(s; q) &= \frac{1}{1-q} [ aq^{2s+1} + (1-q^s)(1-aq^s) - e^{\partial_s} \sqrt{a(1-q^s)(1-aq^s)} q^{s-1/2} \\ &\quad - \sqrt{a(1-q^s)(1-aq^s)} q^{s-1/2} e^{-\partial_s} ]. \end{aligned}$$

Thus  $\mathfrak{H}(s; q)\Phi_n(s; q) = \frac{1-q^n}{1-q}\Phi_n(s; q)$ . This Hamiltonian is factorized,  $\mathfrak{H}(s; q) = \mathfrak{a}^\uparrow(s; q)\mathfrak{a}^\downarrow(s; q)$ , in terms of the difference operators

$$\mathfrak{a}^\downarrow(s; q) := \mathfrak{a}_0^\downarrow(s; q) = \frac{1}{\sqrt{1-q}} [ \sqrt{aq^{s+1/2}} - e^{\partial_s} \sqrt{(1-q^s)(1-aq^s)} ],$$

$$\mathfrak{a}^\uparrow(s; q) := \mathfrak{a}_0^\uparrow(s; q) = \frac{1}{\sqrt{1-q}} \left[ \sqrt{aq^{s+1/2}} - \sqrt{(1-q^s)(1-aq^s)} e^{-\partial_s} \right],$$

that satisfy the commutation relation  $[\mathfrak{a}^\downarrow(s; q), \mathfrak{a}^\uparrow(s; q)]_q = I$ . Thus the dynamical algebra for this family is also  $su_q(1, 1)$ . As in the previous case, the operators  $\mathfrak{a}_2^\downarrow(s; q)$  and  $\mathfrak{a}^\uparrow(s; q)$  are the lowering and raising operators for the functions  $\Phi_n(s; q)$  (see [7]).

A special case of the Al-Salam and Carlitz  $q$ -polynomials of type II are the discrete Hermite  $q$ -polynomials  $\tilde{h}_n(x; q) = i^{-n} V_n^{(-1)}(ix; q)$  of type II. All these cases are closely related to some models of  $q$ -oscillators [10, 12–16].

**4.1.3. Wall polynomials  $p_n(x; a|q)$ .** Our next example is the little  $q$ -Laguerre/Wall polynomials. In this case we define the function

$$\Phi_n(s; a; q) = \frac{1}{d_n(a; q)} \frac{(aq)^{s/2}}{\sqrt{(q; q)_s}} {}_2\phi_1 \left( \begin{matrix} q^{-n}, 0 \\ aq \end{matrix} \middle| q; q^{s+1} \right),$$

where  $x := x(s) = q^s$  and  $A(s) = \sqrt{\nabla x_1(s)}$ .

This function satisfies the discrete orthogonality relation  $\sum_{k=0}^{\infty} \Phi_n(k; q) \Phi_m(k; q) = \delta_{n,m}$ , provided that

$$d_n(a; q) = \frac{(aq)^{n/2}}{(aq; q)_\infty} \sqrt{(q; q)_n (aq^{n+1}; q)_\infty}.$$

In this case  $\sigma(s) = q^{-1}q^s(q^s - 1)$  and  $\sigma(s) + \tau(s)\nabla x_1(s) = -aq^s$ ; thus the corresponding Hamiltonian is

$$\mathfrak{H}(s; a; q) = \frac{1}{(1-q)x} (q(a+1-x)I - \sqrt{aq(1-qx)} e^{\partial_s} - q\sqrt{aq(1-x)} e^{-\partial_s})$$

and  $\mathfrak{H}(s; a; q)\Phi_n(s; a; q) = \frac{1-q^{-n}}{1-q^{-1}}\Phi_n(s; a; q)$ .

Then, the first condition of theorem 3.4 leads to the value  $\alpha = 0$  and  $\zeta = q^{-1/2}$  and the second condition holds if and only if the parameter  $a$  of the above functions is equal to  $q^{-1/2}$ , but they do not lead to the lowering and raising operators.

So to introduce the lowering and raising operators, one has to consider the following operators:

$$\begin{aligned} \mathfrak{a}(s; a; q) &:= \frac{1}{\sqrt{(1-q)x}} (\sqrt{(1-qx)} e^{\partial_s} - \sqrt{aq}I), \\ \mathfrak{a}^\dagger(s; a; q) &:= \frac{1}{\sqrt{(1-q)x}} (\sqrt{q(1-x)} e^{-\partial_s} - \sqrt{aq}I). \end{aligned}$$

The above mutually adjoint operators factorize the Hamiltonian  $\mathfrak{H}(s; a; q)$ , i.e.,  $\mathfrak{H}(s; a; q) = \mathfrak{a}^\dagger(s; a; q)\mathfrak{a}(s; a; q)$ , and they satisfy the commutation relation

$$\mathfrak{a}(s; a/q; q)\mathfrak{a}^\dagger(s; a/q; q) - q^{-1}\mathfrak{a}^\dagger(s; a; q)\mathfrak{a}(s; a; q) = I.$$

From this relation it follows that their action on the functions  $\Phi(s; a; q)$  is given by

$$\begin{aligned} \mathfrak{a}(s; a; q)\Phi_n(s; a; q) &= \sqrt{\frac{1-q^{-n}}{1-q^{-1}}}\Phi_{n-1}(s; aq; q), \\ \mathfrak{a}^\dagger(s; a/q; q)\Phi_n(s; a; q) &= \sqrt{\frac{1-q^{-n-1}}{1-q^{-1}}}\Phi_{n+1}(s; a/q; q). \end{aligned} \tag{27}$$

To verify these formulae one needs to use the property of the normalization constant  $d_n(a; q)$  that

$$d_n(a; q) = \frac{(1-a)q^{n/2}}{\sqrt{a(1-q^{n+1})}} d_{n+1}(a/q; q).$$

Observe that the operators  $a(s; a; q)$  and  $a^\dagger(s; a; q)$  not only lower and raise, respectively, the index  $n$ , but they alter also the parameter  $a$ . That is why an appropriate dynamical algebra in this case is not  $su_q(1, 1)$  (see the discussion above, which follows the proof of theorem 3.4 in section 3). The formulae (27) are equivalent to proving that the forward and backward shift operators for the little  $q$ -Laguerre polynomials have the form (see [7] p 107, (3.20.6) and (3.20.8))

$$\begin{aligned} (e^{\partial_s} - I)p_n(q^s; a; q) &= q^{s+1-n} \frac{1-q^n}{1-aq} p_{n-1}(q^s; aq; q), \\ [(1-q^s)e^{-\partial_s} - aI]p_n(q^s; a; q) &= (1-a)p_{n+1}(q^s; a/q; q), \end{aligned}$$

respectively.

To conclude this example let us mention that the Charlier  $q$ -polynomials  $c_n^{(\mu)}(s; q)$  on the lattice  $x(s) = q^s$ , introduced in [24], are a special case of the Wall polynomials, considered above.

**4.1.4. Discrete Laguerre  $q$ -polynomials  $L_n^{(\alpha)}(x; q)$ .** Let us consider now the Hamiltonian, associated with the discrete Laguerre  $q$ -polynomials. For these polynomials  $\sigma(s) = q^{-1}q^s$  and  $\sigma(s) + \tau(s)\nabla x_1(s) = aq^s(q^s + 1)$  [7]. If we put  $A(s) = \sqrt{\nabla x_1(s)}$  and use the condition (18), then we obtain that  $\alpha = 1$  and  $\zeta = \sqrt{q}$ . Next we use the second condition (19), but the corresponding expression is a constant if and only if  $a = q^{-1/2}$ , i.e., not for any value of the parameter  $a$  the  $q$ -Hamiltonian, associated with the discrete Laguerre  $q$ -polynomials, can be factorized by using the  $\alpha$ -operators, which satisfy the corresponding commutation relation. But as in the previous case, one can define the functions

$$\Phi_n^{(\alpha)}(s; q) = d_n^{-1}(\alpha) \frac{q^{s/2(\alpha+1)}}{\sqrt{(-q^s; q)_\infty}} L_n^{(\alpha)}(x; q), \quad x := x(s) = q^s,$$

where the normalization constant  $d_n(\alpha)$  is given by

$$d_n(\alpha) := \frac{(-q^{\alpha+1}, -q^\alpha; q)_\infty^{1/2}}{(q; q)_\infty} \frac{(q; q)_{n+\alpha}^{1/2}}{q^{n/2}(q; q)_n^{1/2}},$$

and the discrete Laguerre  $q$ -polynomials  $L_n^{(\alpha)}(x; q)$  are

$$L_n^{(\alpha)}(x; q) := \frac{1}{(q; q)_n} {}_2\phi_1 \left( \begin{matrix} q^{-n}, x \\ 0 \end{matrix} \middle| q; q^{n+\alpha+1} \right).$$

These functions satisfy the discrete orthogonality relation  $\sum_{k=0}^{\infty} \Phi_n^{(\alpha)}(k; q) \Phi_m^{(\alpha)}(k; q) = \delta_{n,m}$  and

$$\mathfrak{H}^{(\alpha)}(s; q) \Phi_n^{(\alpha)}(s; q) = \frac{1-q^n}{1-q} \Phi_n^{(\alpha)}(s; q),$$

where the  $q$ -Hamiltonian is a difference operator of the form

$$\mathfrak{H}^{(\alpha)}(s; q) = \frac{1}{1-q} \left\{ [1 + (1+q^{-\alpha})q^{-s}]I - q^{-\frac{s+\alpha}{2}} [\sqrt{1+q^s} e^{\partial_s} + e^{-\partial_s} \sqrt{1+q^s}] q^{-\frac{s}{2}} \right\}.$$

In this case the mutually adjoint operators, that factorize the above  $q$ -Hamiltonian, are

$$\begin{aligned} \mathfrak{a}(s; \alpha; q) &:= \frac{1}{\sqrt{1-q}}(q^{-\frac{\alpha}{2}}I - e^{-\partial_s} \sqrt{1+q^s})q^{-\frac{\alpha}{2}}, \\ \mathfrak{a}^\dagger(s; \alpha; q) &:= \frac{1}{\sqrt{1-q}}q^{-\frac{\alpha}{2}}(q^{-\frac{\alpha}{2}}I - \sqrt{1+q^s} e^{\partial_s}). \end{aligned}$$

They satisfy the commutation relation

$$\mathfrak{a}(s; \alpha - 1; q)\mathfrak{a}^\dagger(s; \alpha - 1; q) - q\mathfrak{a}^\dagger(s; \alpha; q)\mathfrak{a}(s; \alpha; q) = I,$$

from which it follows that

$$\begin{aligned} \mathfrak{a}(s; \alpha; q)\Phi_n^{(\alpha)}(s; q) &= \sqrt{\frac{1-q^n}{1-q}}\Phi_{n-1}^{(\alpha+1)}(s; q), \\ \mathfrak{a}^\dagger(s; \alpha - 1; q)\Phi_n^{(\alpha)}(s; q) &= \sqrt{\frac{1-q^{n+1}}{1-q}}\Phi_{n+1}^{(\alpha-1)}(s; q). \end{aligned}$$

As is the previous case, these relations yield precisely the explicit form of the forward and backward shift operators, respectively, for the  $q$ -Laguerre polynomials  $L_n^{(\alpha)}(q^s; q)$  (see [7], p 109, (3.21.7) and (3.21.9)).

*4.1.5. Other cases in the  $q$ -Hahn tableau.* If one now applies theorems 3.5 and 3.4 to the families of big  $q$ -Jacobi polynomials with  $b = 0$  (big  $q$ -Laguerre), little  $q$ -Jacobi,  $q$ -Meixner,  $q$ -Kravchuk, quantum  $q$ -Kravchuk, affine  $q$ -Kravchuk and alternative  $q$ -Charlier polynomials, then for all these cases it is impossible to solve problem 1. The big and little  $q$ -Jacobi polynomials do not admit a dynamical algebra because the corresponding eigenvalues are not  $q$ -linear functions of  $n$  and in the other cases one of the two conditions of theorem 3.4 fails.

As an example let us consider the case of the  $q$ -Meixner polynomials. For the  $q$ -Meixner polynomials [7], we have  $x(s) = q^{-s}$ ,  $\sigma(s) = c(x(s) - bq)/q$ ,  $\sigma(s) + \tau(s) \nabla x_1(s) = (x(s) - 1)(x(s) + bc)$  and  $\lambda_n = q^{1/2} \frac{1-q^n}{(1-q)^2}$ . Then, the condition (18) gives

$$q^{\frac{1}{2}-\alpha} \sqrt{\frac{q^{-2\alpha+1}(q^\alpha - q^s)(bcq^s + q^\alpha)(q^\alpha - bq^{s+1})}{(q^s - q)(bcq^s + q)(bq^{s+1} - 1)}} = \varsigma.$$

After a careful study of the left-hand side one arrives at the conclusion that it is constant if and only if  $\alpha = 2/3$ ,  $b = q^{-4/3}$ ,  $c = -q^{5/3}$  or  $\alpha = 2/3$ ,  $b = q^{-5/3}$ ,  $c = -q^{4/3}$ . After substituting this in the second condition (19), it becomes clear that the only possibility is the first one, but it corresponds to a non-positive case.

Let us consider this case in more detail. The  $q$ -Meixner polynomials ([7], p 95),

$$M_n(q^{-x}; b, c; q) := {}_2\phi_1 \left( \begin{matrix} q^{-n}, q^{-x} \\ bq \end{matrix} \middle| q; \frac{-q^{n+1}}{c} \right), \quad 0 < b < q^{-1}, \quad c > 0, \quad (28)$$

depend on two parameters  $b$  and  $c$  (in addition to the base  $q$ ) and occupy one level higher in the Askey scheme than  $q$ -Charlier and Al-Salam–Carlitz polynomials of type II. They satisfy a difference equation

$$\begin{aligned} [B(x)(1 - e^{\partial_x}) + D(x)(1 - e^{-\partial_x})]M_n(q^{-x}; b, c; q) &= (1 - q^n)M_n(q^{-x}; b, c; q), \\ B(x) = cq^x(1 - bq^{x+1}), \quad D(x) &= (1 - q^x)(1 + bcq^x). \end{aligned}$$

So  $q$ -Meixner functions, defined as

$$\Phi_n^M(x; b, c; q) := d_n^{-1}(b, c) \left[ \frac{c^x(bq; q)_x}{(q; q)_x(-bcq; q)_x} \right]^{1/2} q^{x(x-1)/4} M_n(q^{-x}; b, c; q),$$



are eigenfunctions of a difference ‘Hamiltonian’  $\mathfrak{H}^M(x; b, c; q)$ ,

$$\mathfrak{H}^M(x; b, c; q)\Phi_n^M(x; b, c; q) = \frac{1 - q^n}{1 - q}\Phi_n^M(x; b, c; q),$$

where

$$\mathfrak{H}^M(x; b, c; q) := \frac{1}{1 - q}[B(x) + D(x) - B^{1/2}(x)e^{\partial_x}D^{1/2}(x) - D^{1/2}(x)e^{-\partial_x}B^{1/2}(x)].$$

The  $q$ -Meixner functions satisfy the discrete orthogonality relation

$$\sum_{k=0}^{\infty} \Phi_m^M(k; b, c; q)\Phi_n^M(k; b, c; q) = \delta_{mn}.$$

One can factorize the ‘Hamiltonian’  $\mathfrak{H}^M(x; b, c; q)$ ,

$$\mathfrak{H}^M(x; b, c; q) = a_M^\uparrow(x; b, c; q)a_M^\downarrow(x; b, c; q),$$

by means of the ‘lowering’ and ‘raising’ difference operators

$$\begin{aligned} a_M^\downarrow(x; b, c; q) &:= \frac{1}{\sqrt{1 - q}}[e^{\partial_x}D^{1/2}(x) - B^{1/2}(x)], \\ a_M^\uparrow(x; b, c; q) &:= \frac{1}{\sqrt{1 - q}}[D^{1/2}(x)e^{-\partial_x} - B^{1/2}(x)]. \end{aligned} \quad (29)$$

The difference operators (29) satisfy a  $q$ -commutation relation of the form

$$a_M^\downarrow(x; b/q, cq; q)a_M^\uparrow(x; b/q, cq; q) - qa_M^\uparrow(x; b, c; q)a_M^\downarrow(x; b, c; q) = I.$$

Their action on the  $q$ -Meixner functions is given by

$$\begin{aligned} a_M^\downarrow(x; b, c; q)\Phi_n^M(x; b, c; q) &= \sqrt{\frac{1 - q^n}{1 - q}}\Phi_{n-1}^M(x; bq, c/q; q), \\ a_M^\uparrow(x; b/q, cq; q)\Phi_n^M(x; b, c; q) &= \sqrt{\frac{1 - q^{n+1}}{1 - q}}\Phi_{n+1}^M(x; b/q, cq; q), \end{aligned} \quad (30)$$

that is, they not only lower and raise, respectively, the index  $n$ , but alter also the parameters  $b$  and  $c$ . The formulae (30) are equivalent to the statement that the forward and backward shift operators for the  $q$ -Meixner polynomials (28) have the form (see [7], p 95, (3.13.2) and (3.13.8))

$$\begin{aligned} (1 - e^{\partial_x})M_n(q^{-x}; b, c; q) &= \frac{1 - q^n}{c(1 - bq)}q^{-x}M_{n-1}(q^{-x}; bq, c/q; q), \\ [cq^x(1 - bq^x) - (1 - q^x)(1 + bcq^x)e^{-\partial_x}]M_n(q^{-x}; b, c; q) \\ &= cq^x(1 - b)M_{n+1}(q^{-x}; b/q, cq; q). \end{aligned}$$

It remains only to remind the reader that when the parameter  $b$  in (28) vanishes, the  $q$ -Meixner polynomials  $M_n(q^{-x}; 0, c; q)$  coincide with the  $q$ -Charlier polynomials ([7], p 112)

$$C_n(q^{-x}; c; q) := {}_2\phi_1\left(q^{-n}, q^{-x} \middle| q; \frac{-q^{n+1}}{c}\right). \quad (31)$$

In this case  $B(x) = cq^x$  and  $D(x) = 1 - q^x$ . The appropriate formulae for the  $q$ -Charlier polynomials (31) are therefore easy consequences of the corresponding formulae for the  $q$ -Meixner polynomials (28) with the vanishing value of the parameter  $b$ .

4.2. Askey–Wilson cases

Using the linearity of the difference equation (1) we can consider, without loss of generality, the following lattice  $x(s) = \frac{1}{2}(q^s + q^{-s})$ , for which  $\mu = 0$ . Then,

$$\sigma(s) = Cq^{-2s} \prod_{i=1}^4 (q^s - q^{s_i}) = q^{-2s} \prod_{i=1}^4 (q^s - z_i), \quad \sigma(-s - \mu) = Cq^{2s} \prod_{i=1}^4 (q^{-s} - z_i).$$

It is well known that the general case when the zeros of  $\tilde{\sigma}$ , namely  $z_1 z_2 z_3 z_4 \neq 0$ , corresponds to the Askey–Wilson polynomials [7]. We will use theorem 3.4 to solve the factorization problem for the whole  $q$ -Askey tableau [7] and Nikiforov–Uvarov tableau [4, 5].

The Askey–Wilson polynomials on the lattice  $x(s) = \frac{1}{2}(q^s + q^{-s}) = \cos \theta$  where  $q^s = e^{i\theta}$ , defined by [7]

$$p_n(x(s); a, b, c, d|q) = \frac{(ab, ac, ad; q)_n}{a^n} {}_4\phi_3 \left( \begin{matrix} q^{-n}, q^{n-1}abcd, aq^{-s}, aq^s \\ ab, ac, ad \end{matrix} \middle| q; q \right),$$

where  $a = z_1, b = z_2, c = z_3, d = z_4$  and  $\mu = 0$ . Their orthogonality relation is of the form

$$\int_{-1}^1 \omega(x) p_n(x; a, b, c, d) p_m(x; a, b, c, d) dx = \delta_{nm} d_n^2,$$

where

$$\omega(x) = \frac{h(x, 1)h(x, -1)h(x, q^{\frac{1}{2}})h(x, -q^{\frac{1}{2}})}{2\pi\sqrt{1-x^2}h(x, a)h(x, b)h(x, c)h(x, d)}, \quad h(x, \alpha) = \prod_{k=0}^{\infty} [1 - 2\alpha x q^k + \alpha^2 q^{2k}],$$

and the norm is given by

$$d_n^2 = \frac{(abcdq^{n-1}, abcdq^{2n}; q)_{\infty}}{(q^{n+1}, abq^n, acq^n, adq^n, bcq^n, bdq^n, cdq^n; q)_{\infty}}.$$

The Askey–Wilson functions can be defined by

$$\Phi_n(s; q) = \frac{\sqrt{\omega(s)}A(s)}{d_n} p_n(x(s); a, b, c, d), \quad x(s) = \cos \theta, \quad q^s = e^{i\theta}.$$

Taking  $A(s) = \sqrt{\nabla x_1(s)}$ , we have the orthogonality  $\int_{-1}^1 \Phi_n(s; q)\Phi_m(s; q)/\nabla x_1(s) dx = \delta_{n,m}$ , and  $\mathfrak{H}(s; q)\Phi_n(s; q) = \lambda_n \Phi_n(s; q)$ , where  $\lambda_n = q(q^{-n} - 1)(1 - abcdq^{n-1})$ , and  $\mathfrak{H}(s; q)$  is given by (9),  $\sigma(s) = C_{\sigma} q^{-2s} (q^s - a)(q^s - b)(q^s - c)(q^s - d)$  and  $\mu = 0$ . Thus the Hamiltonian, associated with these Askey–Wilson functions, is

$$\begin{aligned} \mathfrak{H}(s; q) = & \frac{-1}{k_q^2 \sqrt{\sin \theta}} \left( \frac{\sqrt{\sigma(s)\sigma(-s+1)}}{\sin(\theta + \frac{i}{2} \log q) \sqrt{\sin(\theta + i \log q)}} e^{-\partial_s} \right. \\ & \left. + \frac{\sqrt{\sigma(s+1)\sigma(-s)}}{\sin(\theta - \frac{i}{2} \log q) \sqrt{\sin(\theta - i \log q)}} e^{\partial_s} \right) \\ & + \frac{1}{k_q^2 \sin \theta} \left( \frac{\sigma(s)}{\sin(\theta + \frac{i}{2} \log q)} + \frac{\sigma(-s)}{\sin(\theta - \frac{i}{2} \log q)} \right) I. \end{aligned}$$

If we now use remark 3.6, then the first condition of theorem 3.4 holds for  $\alpha = \frac{1}{2}$ . In fact, a straightforward calculations shows that

$$\Delta x_1(s + \gamma) = \frac{k_q}{2} q^{-s-\gamma} (q^{s+\gamma} + 1)(q^{s+\gamma} - 1).$$

Hence, the condition (18) has the form ( $e^{i\theta} = q^s$ )

$$q^{-2\alpha+1} \left( \frac{q^{s-\frac{1}{2}} - 1}{q^{s-\alpha} - 1} \sqrt{\frac{(q^{s-1} - 1)(q^s - 1)}{(q^{s-\alpha-\frac{1}{2}} - 1)(q^{s-\alpha+\frac{1}{2}} - 1)}} \sqrt{\prod_{i=1}^4 \frac{(q^{s-\alpha} - z_i)(q^{-s+\alpha} - z_i)}{(q^s - z_i)(q^{-s+1} - z_i)}} \right) \\ \times \frac{q^{s-\frac{1}{2}} + 1}{q^{s-\alpha} + 1} \sqrt{\frac{(q^{s-1} + 1)(q^s + 1)}{(q^{s-\alpha-\frac{1}{2}} + 1)(q^{s-\alpha+\frac{1}{2}} + 1)}} = \zeta.$$

If we look at the expression in the second line, we find that it is a constant if and only if  $\alpha = \frac{1}{2}$ , and thus the first condition transforms into

$$\sqrt{\frac{\prod_{i=1}^4 (q^{s-1/2} - z_i)(q^{-s+1/2} - z_i)}{\prod_{i=1}^4 (q^s - z_i)(q^{-s+1} - z_i)}} = \zeta. \quad (32)$$

Then, the simplest case for which the condition (18) holds is when  $z_1 z_2 = q^{\frac{1}{2}}$  and  $z_3 z_4 = q^{\frac{1}{2}}$  (with the corresponding permutations of the roots). In this case  $\zeta = 1$ . Since  $\sigma(s) = \sigma(-s + \frac{1}{2})$ , the second condition gives

$$\frac{1}{\nabla x_1(s)} \left( \frac{\sigma(s + \frac{1}{2})}{\Delta x(s)} + \frac{\sigma(-s + \frac{1}{2})}{\nabla x(s)} - \frac{\sigma(s)}{\nabla x(s)} - \frac{\sigma(-s)}{\Delta x(s)} \right) = 0.$$

Thus, we have  $a_{1/2}^\uparrow(s; q) a_{1/2}^\downarrow(s) = \mathfrak{H}(s; q)$  and  $[a_{1/2}^\downarrow(s; q), a_{1/2}^\uparrow(s)]_q = 0$ , where

$$a_{1/2}^\downarrow(s; q) = e^{\frac{1}{2}\partial_s} \sqrt{\frac{\sigma(s)}{-k_q^2 \sin \theta \sin(\theta + \frac{1}{2} \log q)}} - e^{-\frac{1}{2}\partial_s} \sqrt{\frac{\sigma(-s)}{-k_q^2 \sin \theta \sin(\theta - \frac{1}{2} \log q)}}, \\ a_{1/2}^\uparrow(s; q) = \sqrt{\frac{\sigma(s)}{-k_q^2 \sin \theta \sin(\theta + \frac{1}{2} \log q)}} e^{-\frac{1}{2}\partial_s} - \sqrt{\frac{\sigma(-s)}{-k_q^2 \sin \theta \sin(\theta - \frac{1}{2} \log q)}} e^{\frac{1}{2}\partial_s}.$$

Since the  $\alpha$ -operators are commuting, this case is not so interesting in applications (e.g., for  $q$ -models of the harmonic oscillators). A special case of the Askey–Wilson polynomials are the continuous  $q$ -Jacobi polynomials corresponding to the roots  $a = q^{\alpha'/2+1/4}$ ,  $b = q^{\alpha'/2+3/4}$ ,  $c = -q^{\beta'/2+1/4}$ ,  $d = -q^{\beta'/2+3/4}$  [7], then we can solve problem 1 only in the case when  $\alpha' = \beta' = -1/2$ .

The next case is when one of the roots  $z_i$  vanishes. This is the 0-Askey–Wilson polynomials or the continuous dual  $q$ -Hahn [7]. In this case the first condition (18) (equivalently (32) holds only when

$$(z_1, z_2, z_3) = (t, \frac{1}{2} - t, \frac{1}{4}), \quad t \in \mathbb{R}.$$

With the above choice of  $z_i$ , it is impossible to fulfil the second condition (19); hence it is impossible to obtain a simple closed dynamical algebra.

**4.2.1. Continuous  $q$ -Laguerre polynomials.** Let us now consider the case when the Askey–Wilson polynomials have two parameters equal to zero. In order that the condition (18) take place, the other two non-vanishing parameters should satisfy that their product is equal to  $q^{\frac{1}{2}}$ . Under this condition  $\zeta = q^{-\frac{1}{2}}$ . Then the second condition yields  $\Lambda = \frac{4C_\sigma(\sqrt{q}-1)}{k_q^2}$ . Then, the Askey–Wilson Hamiltonian with two zero roots of  $\sigma$  admits a factorization with a non-trivial dynamical algebra. An example of this family is the continuous  $q$ -Laguerre

polynomials  $P_n^{(a)}(x|q)$  [7],  $x(s) = \cos \theta$ , when  $a = -1/2$  for which the Hamiltonian has the form  $(A(s) = 1)$

$$\begin{aligned} \mathfrak{H}(s; q) = & -\frac{C_\sigma}{k_q \sin \theta} \left( \frac{\sqrt{(q^{s+1} - 1)(q^{s+1} - q^{\frac{1}{2}})(q^{-s} - 1)(q^{-s} - q^{\frac{1}{2}})}}{\sin(\theta + \frac{1}{2} \log q)} e^{-\partial_s} \right. \\ & \left. + \frac{\sqrt{(q^s - 1)(q^s - q^{\frac{1}{2}})(q^{1-s} - 1)(q^{1-s} - q^{\frac{1}{2}})}}{\sin(\theta - \frac{1}{2} \log q)} e^{\partial_s} \right) \\ & - \frac{C_\sigma}{k_q \sin \theta} \left( \frac{(q^s - 1)(q^s - q^{\frac{1}{2}})}{\sin(\theta + \frac{1}{2} \log q)} + \frac{(q^{-s} - 1)(q^{-s} - q^{\frac{1}{2}})}{\sin(\theta - \frac{1}{2} \log q)} \right) I, \end{aligned}$$

and

$$\begin{aligned} a_{1/2}^\downarrow(s; q) &= \frac{\sqrt{-C_\sigma}}{k_q \sin \theta} \left( e^{\frac{1}{2}\partial_s} \sqrt{(q^s - 1)(q^s - q^{\frac{1}{2}})} - e^{-\frac{1}{2}\partial_s} \sqrt{(q^{-s} - 1)(q^{-s} - q^{\frac{1}{2}})} \right), \\ a_{1/2}^\uparrow(s; q) &= \frac{\sqrt{-C_\sigma}}{k_q \sin \theta} \left( \sqrt{(q^s - 1)(q^s - q^{\frac{1}{2}})} e^{-\frac{1}{2}\partial_s} - \sqrt{(q^{-s} - 1)(q^{-s} - q^{\frac{1}{2}})} e^{\frac{1}{2}\partial_s} \right). \end{aligned}$$

Then for the functions

$$\Phi_n(x(s)) = \sqrt{\frac{(q^{\frac{1}{2}}; q)_n (q, q^{\frac{1}{2}}; q)_\infty \omega(s)}{(q; q)_n}} {}_3\phi_2 \left( \begin{matrix} q^{-n}, q^{-s}, q^s \\ q^{\frac{1}{2}}, 0 \end{matrix} \middle| q; q \right), \quad q^s = e^{i\theta}, \quad x(s) = \cos \theta,$$

we have the orthogonality  $\int_{-1}^1 \Phi_n(x; q) \Phi_m(x; q) dx = \delta_{n,m}$ , and

$$\mathfrak{H}(s; q) \Phi_n(s; q) = q(q^{-n} - 1) \Phi_n(s; q), \quad \mathfrak{H}(s; q) = a_{1/2}^\uparrow(s; q) a_{1/2}^\downarrow(s),$$

and

$$[a_{1/2}^\downarrow(s; q), a_{1/2}^\uparrow(s)]_{q^{-1/2}} = \frac{4C_\sigma(\sqrt{q} - 1)}{k_q^2}.$$

Thus choosing  $C_\sigma = -\frac{k_q^2}{4(1-\sqrt{q})}$  we obtain the relation  $[a_{1/2}^\downarrow(s; q), a_{1/2}^\uparrow(s)]_{q^{-1/2}} = I$ .

The case of Askey–Wilson polynomials with three zero parameters is analogous to the case of one zero parameter and it is not possible to solve problem 1. An example of this case are the continuous big  $q$ -Hermite polynomials [7].

**4.2.2. Continuous  $q$ -Hermite polynomials.** Finally, if one takes the Askey–Wilson polynomials with vanishing parameters  $a, b, c, d$ , this gives the continuous  $q$ -Hermite polynomials [7]. In this case  $\sigma(z) = C_\sigma q^{2z}$ .

Let choose  $A(s) = \sqrt{\nabla x_1(s)}$ . Taking into account that this family is a special case of the Askey–Wilson polynomials when all parameters  $a = b = c = d = 0$ , one directly obtains, by using  $\zeta = 1/q$ , that in this case the Hamiltonian is given by

$$\begin{aligned} \mathfrak{H}(s; q) = & \frac{-1}{k_q^2 \sqrt{\sin \theta}} \left( \frac{C_\sigma q}{\sin(\theta + \frac{1}{2} \log q) \sqrt{\sin(\theta + i \log q)}} e^{-\partial_s} \right. \\ & \left. + \frac{C_\sigma q}{\sin(\theta - \frac{1}{2} \log q) \sqrt{\sin(\theta - i \log q)}} e^{\partial_s} \right) \\ & + \frac{1}{k_q^2 \sin \theta} \left( \frac{C_\sigma q^{2s}}{\sin(\theta + \frac{1}{2} \log q)} + \frac{C_\sigma q^{-2s}}{\sin(\theta - \frac{1}{2} \log q)} \right) I, \end{aligned}$$

and the  $\alpha$ -operators

$$a_{1/2}^\downarrow(s; q) = e^{\frac{1}{2}\partial_s} \sqrt{\frac{C_\sigma q^{2s}}{-k_q^2 \sin \theta \sin(\theta + \frac{1}{2} \log q)}} - e^{-\frac{1}{2}\partial_s} \sqrt{\frac{C_\sigma q^{-2s}}{-k_q^2 \sin \theta \sin(\theta - \frac{1}{2} \log q)}}$$

$$a_{1/2}^\uparrow(s; q) = \sqrt{\frac{C_\sigma q^{2s}}{-k_q^2 \sin \theta \sin(\theta + \frac{1}{2} \log q)}} e^{-\frac{1}{2}\partial_s} - \sqrt{\frac{C_\sigma q^{-2s}}{-k_q^2 \sin \theta \sin(\theta - \frac{1}{2} \log q)}} e^{\frac{1}{2}\partial_s},$$

are such that

$$a^\uparrow(s; q) a^\downarrow(s; q) = \mathfrak{H}(s; q) \quad \text{and} \quad [a^\downarrow(s; q), a^\uparrow(s; q)]_{1/q} = \frac{4C_\sigma}{k_q}.$$

Note that for getting the normalized commutation relations it is sufficient to choose  $C_\sigma = k_q/4$ .

Another possible choice is  $A(s) = 1$  [12], hence a straightforward calculus shows that the two conditions in theorem 3.4 are true if  $\zeta = q^{-1}$ , thus  $\Lambda = 4C_\sigma k_q^{-1}$ . With this choice the orthogonality of the functions  $\Phi_n$  is  $\int_{-1}^1 \Phi_n(s; q) \Phi_m(s; q) dx = \delta_{n,m}$ . In this case, the Hamiltonian is equal to

$$\mathfrak{H}(s; q) = \frac{C_\sigma q}{k_q^2} \left\{ \frac{e^{-\partial_s}}{\sin \theta \sin(\theta + \frac{1}{2} \ln q)} + \frac{e^{\partial_s}}{\sin(\theta - \frac{1}{2} \ln q) \sin \theta} - \frac{4}{\sqrt{q}} \left( 1 - \frac{1+q}{q+q^{-1}-2\cos 2\theta} \right) I \right\},$$

and

$$\alpha^\downarrow(s; q) := a_{1/2}^\downarrow(s) = \frac{\sqrt{-C_\sigma}}{k_q \sin \theta} (e^{\frac{1}{2}\partial_s} q^s - e^{-\frac{1}{2}\partial_s} q^{-s}),$$

$$\alpha^\uparrow(s; q) := a_{1/2}^\uparrow(s) = \frac{\sqrt{-C_\sigma}}{k_q \sin \theta} (q^s e^{-\frac{1}{2}\partial_s} - q^{-s} e^{\frac{1}{2}\partial_s}).$$

In terms of these operators

$$\mathfrak{H}(s; q) = \alpha^\uparrow(s; q) \alpha^\downarrow(s; q) \quad \text{and} \quad [a^\downarrow(s; q), a^\uparrow(s; q)]_{q^{-1}} = \frac{4C_\sigma}{k_q}.$$

As before we now can choose  $C_\sigma = k_q/4$ . This case was first considered in [13], see also [25].

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## Appendix

Here we present a simple proof of theorem 3.5 when  $\alpha = 0$ . Note that in this case  $b(s; q) = a^\dagger(s; q)$ , i.e., the operator  $b$  is the adjoint of  $a$ .

**Theorem 3.5a.** Let  $\{\Phi_n(s; q)\}$  be the eigenfunctions of the operator  $\mathfrak{H}(s; q)$ , corresponding to the eigenvalues  $\{\lambda_n(q)\}$ . Suppose that the  $\mathfrak{H}(s; q)$  admits the factorization (13). If the operators  $a(s; q)$  and  $b(s; q)$  in (13) satisfy the  $q$ -commutation relation  $[a(s; q), b(s; q)]_q = I$ , then the eigenvalues  $\lambda_n(q)$  are  $q$ -linear or  $q^{-1}$ -linear functions of  $n$ , i.e., either  $\lambda_n(q) = C_1 q^n + C_3$  or  $\lambda_n(q) = C_2 q^{-n} + C_3$ , respectively.

**Proof.** Obviously, the operator  $\mathfrak{H}(s; q)$  is diagonal in the basis consisting of its eigenfunctions  $\Phi_n(s; q)$ . By hypothesis, this operator admits at the same time the factorization (13) in terms of  $a(s; q)$  and  $b(s; q)$ . But according to proposition 3.2 the function  $a(s; q)\Phi_n(s; q)$  is also the eigenfunction of  $\mathfrak{H}(s; q)$ , associated with the eigenvalue  $q^{-1}[\lambda_n(q) - 1]$ . Hence the  $a(s; q)$  is either the lowering operator or the raising operator. In the former case this means that

$$q^{-1}[\lambda_n(q) - 1] = \lambda_{n-1}(q) + C,$$

from which it follows that  $C_2 = 0$  (i.e., the spectrum  $\{\lambda_n(q)\}$  is a  $q$ -linear one) and  $C = q^{-1}[(1 - q)C_3 - 1]$ . In latter case the corresponding relation is

$$q^{-1}[\lambda_n(q) - 1] = \lambda_{n+1}(q) + C,$$

which holds when  $C_1 = 0$  (i.e., the spectrum is  $q^{-1}$ -linear) and  $C = q^{-1}[(1 - q)C_3 - 1]$ . The proof of the theorem is thus complete.  $\square$

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